GLOBAL EDITION

Linear Algebra and Its Applications

SIXTH EDITION

David C. Lay • Steven R. Lay • Judi J. McDonald

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and Its Applications

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To my wife, Lillian, and our children, Christina, Deborah, and Melissa, whose support, encouragement, and faithful prayers made this book possible.

David C. Lay

About the Authors

David C. Lay

As a founding member of the NSF-sponsored Linear Algebra Curriculum Study Group (LACSG), David Lay was a leader in the movement to modernize the linear algebra curriculum and shared those ideas with students and faculty through his authorship of the first four editions of this textbook. David C. Lay earned a B.A. from Aurora University (Illinois), and an M.A. and Ph.D. from the University of California at Los Angeles. David Lay was an educator and research mathematician for more than 40 years, mostly at the University of Maryland, College Park. He also served as a visiting professor at the University of Amsterdam, the Free University in Amsterdam, and the University of Kaiserslautern, Germany. He published more than 30 research articles on functional analysis and linear algebra. Lay was also a coauthor of several mathematics texts, including *Introduction to Functional Analysis* with Angus E. Taylor, *Calculus and Its Applications*, with L. J. Goldstein and D. I. Schneider, and *Linear Algebra Gems—Assets for Undergraduate Mathematics*, with D. Carlson, C. R. Johnson, and A. D. Porter.

David Lay received four university awards for teaching excellence, including, in 1996, the title of Distinguished Scholar-Teacher of the University of Maryland. In 1994, he was given one of the Mathematical Association of America's Awards for Distinguished College or University Teaching of Mathematics. He was elected by the university students to membership in Alpha Lambda Delta National Scholastic Honor Society and Golden Key National Honor Society. In 1989, Aurora University conferred on him the Outstanding Alumnus award. David Lay was a member of the American Mathematical Society, the Canadian Mathematical Society, the International Linear Algebra Society, the Mathematical Association of America, Sigma Xi, and the Society for Industrial and Applied Mathematics. He also served several terms on the national board of the Association of Christians in the Mathematical Sciences.

In October 2018, David Lay passed away, but his legacy continues to benefit students of linear algebra as they study the subject in this widely acclaimed text.

Steven R. Lay

Steven R. Lay began his teaching career at Aurora University (Illinois) in 1971, after earning an M.A. and a Ph.D. in mathematics from the University of California at Los Angeles. His career in mathematics was interrupted for eight years while serving as a missionary in Japan. Upon his return to the States in 1998, he joined the mathematics faculty at Lee University (Tennessee) and has been there ever since. Since then he has supported his brother David in refining and expanding the scope of this popular linear algebra text, including writing most of Chapters 8 and 9. Steven is also the author of three college-level mathematics texts: *Convex Sets and Their Applications, Analysis with an Introduction to Proof, and Principles of Algebra*.

In 1985, Steven received the Excellence in Teaching Award at Aurora University. He and David, and their father, Dr. L. Clark Lay, are all distinguished mathematicians, and in 1989, they jointly received the Outstanding Alumnus award from their alma mater, Aurora University. In 2006, Steven was honored to receive the Excellence in Scholarship Award at Lee University. He is a member of the American Mathematical Society, the Mathematics Association of America, and the Association of Christians in the Mathematical Sciences.

Judi J. McDonald

Judi J. McDonald became a co-author on the fifth edition, having worked closely with David on the fourth edition. She holds a B.Sc. in Mathematics from the University of Alberta, and an M.A. and Ph.D. from the University of Wisconsin. As a professor of Mathematics, she has more than 40 publications in linear algebra research journals and more than 20 students have completed graduate degrees in linear algebra under her supervision. She is an associate dean of the Graduate School at Washington State University and a former chair of the Faculty Senate. She has worked with the mathematics outreach project Math Central (http://mathcentral.uregina.ca/) and is a member of the second Linear Algebra Curriculum Study Group (LACSG 2.0).

Judi has received three teaching awards: two Inspiring Teaching awards at the University of Regina, and the Thomas Lutz College of Arts and Sciences Teaching Award at Washington State University. She also received the College of Arts and Sciences Institutional Service Award at Washington State University. Throughout her career, she has been an active member of the International Linear Algebra Society and the Association for Women in Mathematics. She has also been a member of the Canadian Mathematical Society, the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

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Preface

The response of students and teachers to the first five editions of *Linear Algebra and Its Applications* has been most gratifying. This *Sixth Edition* provides substantial support both for teaching and for using technology in the course. As before, the text provides a modern elementary introduction to linear algebra and a broad selection of interesting classical and leading-edge applications. The material is accessible to students with the maturity that should come from successful completion of two semesters of college-level mathematics, usually calculus.

The main goal of the text is to help students master the basic concepts and skills they will use later in their careers. The topics here follow the recommendations of the original Linear Algebra Curriculum Study Group (LACSG), which were based on a careful investigation of the real needs of the students and a consensus among professionals in many disciplines that use linear algebra. Ideas being discussed by the second Linear Algebra Curriculum Study Group (LACSG 2.0) have also been included. We hope this course will be one of the most useful and interesting mathematics classes taken by undergraduates.

What's New in This Edition

The *Sixth Edition* has exciting new material, examples, and online resources. After talking with high-tech industry researchers and colleagues in applied areas, we added new topics, vignettes, and applications with the intention of highlighting for students and faculty the linear algebraic foundational material for machine learning, artificial intelligence, data science, and digital signal processing.

Content Changes

- Since matrix multiplication is a highly useful skill, we added new examples in Chapter 2 to show how matrix multiplication is used to identify patterns and scrub data. Corresponding exercises have been created to allow students to explore using matrix multiplication in various ways.
- In our conversations with colleagues in industry and electrical engineering, we heard repeatedly how important understanding abstract vector spaces is to their work. After reading the reviewers' comments for Chapter 4, we reorganized the chapter, condensing some of the material on column, row, and null spaces; moving Markov chains to the end of Chapter 5; and creating a new section on signal processing. We view signals

as an infinite dimensional vector space and illustrate the usefulness of linear transformations to filter out unwanted "vectors" (a.k.a. noise), analyze data, and enhance signals.

- By moving Markov chains to the end of Chapter 5, we can now discuss the steady state vector as an eigenvector. We also reorganized some of the summary material on determinants and change of basis to be more specific to the way they are used in this chapter.
- In Chapter 6, we present pattern recognition as an application of orthogonality, and the section on linear models now illustrates how machine learning relates to curve fitting.
- Chapter 9 on optimization was previously available only as an online file. It has now been moved into the regular textbook where it is more readily available to faculty and students. After an opening section on finding optimal strategies to two-person zero-sum games, the rest of the chapter presents an introduction to linear programming—from two-dimensional problems that can be solved geometrically to higher dimensional problems that are solved using the Simplex Method.

Other Changes

- In the high-tech industry, where most computations are done on computers, judging the validity of information and computations is an important step in preparing and analyzing data. In this edition, students are encouraged to learn to analyze their own computations to see if they are consistent with the data at hand and the questions being asked. For this reason, we have added "Reasonable Answers" advice and exercises to guide students.
- We have added a list of projects to the end of each chapter (available online and in MyLab Math). Some of these projects were previously available online and have a wide range of themes from using linear transformations to create art to exploring additional ideas in mathematics. They can be used for group work or to enhance the learning of individual students.
- PowerPoint lecture slides have been updated to cover all sections of the text and cover them more thoroughly.

Distinctive Features

Early Introduction of Key Concepts

Many fundamental ideas of linear algebra are introduced within the first seven lectures, in the concrete setting of $\mathbb{R}^n$, and then gradually examined from different points of view. Later generalizations of these concepts appear as natural extensions of familiar ideas, visualized through the geometric intuition developed in Chapter 1. A major achievement of this text is that the level of difficulty is fairly even throughout the course.

A Modern View of Matrix Multiplication

Good notation is crucial, and the text reflects the way scientists and engineers actually use linear algebra in practice. The definitions and proofs focus on the columns of a matrix rather than on the matrix entries. A central theme is to view a matrix–vector product $A\mathbf{x}$ as a linear combination of the columns of A. This modern approach simplifies many arguments, and it ties vector space ideas into the study of linear systems.

Linear Transformations

Linear transformations form a "thread" that is woven into the fabric of the text. Their use enhances the geometric flavor of the text. In Chapter 1, for instance, linear transformations provide a dynamic and graphical view of matrix–vector multiplication.

Eigenvalues and Dynamical Systems

Eigenvalues appear fairly early in the text, in Chapters 5 and 7. Because this material is spread over several weeks, students have more time than usual to absorb and review these critical concepts. Eigenvalues are motivated by and applied to discrete and continuous dynamical systems, which appear in Sections 1.10, 4.8, and 5.9, and in five sections of Chapter 5. Some courses reach Chapter 5 after about five weeks by covering Sections 2.8 and 2.9 instead of Chapter 4. These two optional sections present all the vector space concepts from Chapter 4 needed for Chapter 5.

Orthogonality and Least-Squares Problems

These topics receive a more comprehensive treatment than is commonly found in beginning texts. The original Linear Algebra Curriculum Study Group has emphasized the need for a substantial unit on orthogonality and least-squares problems, because orthogonality plays such an important role in computer calculations and numerical linear algebra and because inconsistent linear systems arise so often in practical work.

Pedagogical Features

Applications

A broad selection of applications illustrates the power of linear algebra to explain fundamental principles and simplify calculations in engineering, computer science, mathematics, physics, biology, economics, and statistics. Some applications appear in separate sections; others are treated in examples and exercises. In addition, each chapter opens with an introductory vignette that sets the stage for some application of linear algebra and provides a motivation for developing the mathematics that follows.

A Strong Geometric Emphasis

Every major concept in the course is given a geometric interpretation, because many students learn better when they can visualize an idea. There are substantially more drawings here than usual, and some of the figures have never before appeared in a linear algebra text. Interactive versions of many of these figures appear in MyLab Math.

Examples

This text devotes a larger proportion of its expository material to examples than do most linear algebra texts. There are more examples than an instructor would ordinarily present in class. But because the examples are written carefully, with lots of detail, students can read them on their own.

Theorems and Proofs

Important results are stated as theorems. Other useful facts are displayed in tinted boxes, for easy reference. Most of the theorems have formal proofs, written with the beginner student in mind. In a few cases, the essential calculations of a proof are exhibited in a carefully chosen example. Some routine verifications are saved for exercises, when they will benefit students.

Practice Problems

A few carefully selected Practice Problems appear just before each exercise set. Complete solutions follow the exercise set. These problems either focus on potential trouble spots in the exercise set or provide a "warm-up" for the exercises, and the solutions often contain helpful hints or warnings about the homework.

Exercises

The abundant supply of exercises ranges from routine computations to conceptual questions that require more thought. A good number of innovative questions pinpoint conceptual difficulties that we have found on student papers over the years. Each exercise set is carefully arranged in the same general order as the text; homework assignments are readily available when only part of a section is discussed. A notable feature of the exercises is their numerical simplicity. Problems "unfold" quickly, so students spend little time on numerical calculations. The exercises concentrate on teaching understanding rather than mechanical calculations. The exercises in the *Sixth Edition* maintain the integrity of the exercises from previous editions, while providing fresh problems for students and instructors.

Exercises marked with the symbol **T** are designed to be worked with the aid of a "matrix program" (a computer program, such as MATLAB, Maple, Mathematica, MathCad, or Derive, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments).

True/False Questions

To encourage students to read all of the text and to think critically, we have developed over 300 simple true/false questions that appear throughout the text, just after the computational problems. They can be answered directly from the text, and they prepare students for the conceptual problems that follow. Students appreciate these questions-after they get used to the importance of reading the text carefully. Based on class testing and discussions with students, we decided not to put the answers in the text. (The *Study Guide*, in MyLab Math, tells the students where to find the answers to the odd-numbered questions.) An additional 150 true/false questions (mostly at the ends of chapters) test understanding of the material. The text does provide simple T/F answers to most of these supplementary exercises, but it omits the justifications for the answers (which usually require some thought).

Writing Exercises

An ability to write coherent mathematical statements in English is essential for all students of linear algebra, not just those who may go to graduate school in mathematics.

The text includes many exercises for which a written justification is part of the answer. Conceptual exercises that require a short proof usually contain hints that help a student get started. For all odd-numbered writing exercises, either a solution is included at the back of the text or a hint is provided and the solution is given in the *Study Guide*.

Projects

A list of projects (available online) have been identified at the end of each chapter. They can be used by individual students or in groups. These projects provide the opportunity for students to explore fundamental concepts and applications in more detail. Two of the projects even encourage students to engage their creative side and use linear transformations to build artwork.

Reasonable Answers

Many of our students will enter a workforce where important decisions are being made based on answers provided by computers and other machines. The Reasonable Answers boxes and exercises help students develop an awareness of the need to analyze their answers for correctness and accuracy.

Computational Topics

The text stresses the impact of the computer on both the development and practice of linear algebra in science and engineering. Frequent Numerical Notes draw attention to issues in computing and distinguish between theoretical concepts, such as matrix inversion, and computer implementations, such as LU factorizations.

Acknowledgments

David Lay was grateful to many people who helped him over the years with various aspects of this book. He was particularly grateful to Israel Gohberg and Robert Ellis for more than fifteen years of research collaboration, which greatly shaped his view of linear algebra. And he was privileged to be a member of the Linear Algebra Curriculum Study Group along with David Carlson, Charles Johnson, and Duane Porter. Their creative ideas about teaching linear algebra have influenced this text in significant ways. He often spoke fondly of three good friends who guided the development of the book nearly from the beginning—giving wise counsel and encouragement—Greg Tobin, publisher; Laurie Rosatone, former editor; and William Hoffman, former editor.

Judi and Steven have been privileged to work on recent editions of Professor David Lay's linear algebra book. In making this revision, we have attempted to maintain the basic approach and the clarity of style that has made earlier editions popular with students and faculty. We thank Eric Schulz for sharing his considerable technological and pedagogical expertise in the creation of the electronic textbook. His help and encouragement were essential in bringing the figures and examples to life in the Wolfram Cloud version of this textbook.

Mathew Hudelson has been a valuable colleague in preparing the *Sixth Edition;* he is always willing to brainstorm about concepts or ideas and test out new writing and exercises. He contributed the idea for new vignette for Chapter 3 and the accompanying project. He has helped with new exercises throughout the text. Harley Weston has provided Judi with many years of good conversations about how, why, and who we appeal to when we present mathematical material in different ways. Katerina Tsatsomeros' artistic side has been a definite asset when we needed artwork to transform (the fish and the sheep), improved writing in the new introductory vignettes, or information from the perspective of college-age students.

We appreciate the encouragement and shared expertise from Nella Ludlow, Thomas Fischer, Amy Johnston, Cassandra Seubert, and Mike Manzano. They provided information about important applications of linear algebra and ideas for new examples and exercises. In particular, the new vignettes and material in Chapters 4 and 6 were inspired by conversations with these individuals.

We are energized by Sepideh Stewart and the other new Linear Algebra Curriculum Study Group (LACSG 2.0) members: Sheldon Axler, Rob Beezer, Eugene Boman, Minerva Catral, Guershon Harel, David Strong, and Megan Wawro. Initial meetings of this group have provided valuable guidance in revising the *Sixth Edition.*

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We thank Jeff Weidenaar, our content manager, for his continued careful, well-thought-out advice. Project Manager Ron Hampton has been a tremendous help guiding us through the production process. We are also grateful to Stacey Sveum and Rosemary Morton, our marketers, and Jon Krebs, our editorial associate, who have also contributed to the success of this edition.

Steven R. Lay and Judi J. McDonald

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Lay, Lay, McDonald

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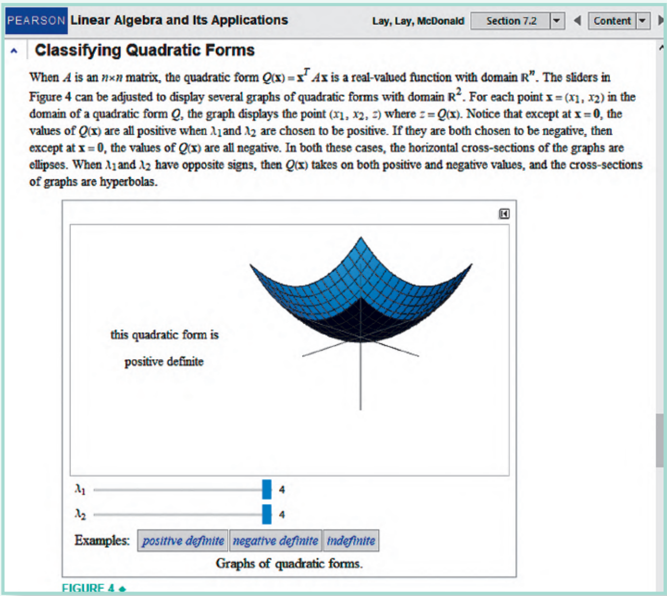

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Interactive Figure files are available within MyLab Math to use as a teaching tool for classroom demonstrations. Instructors can illustrate concepts that are difficult for students to visualize, leading to greater conceptual understanding.

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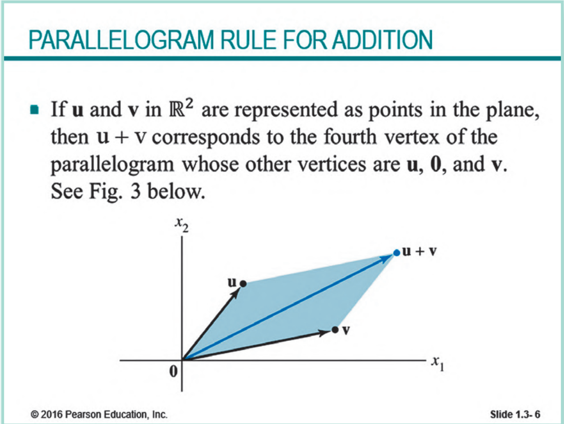
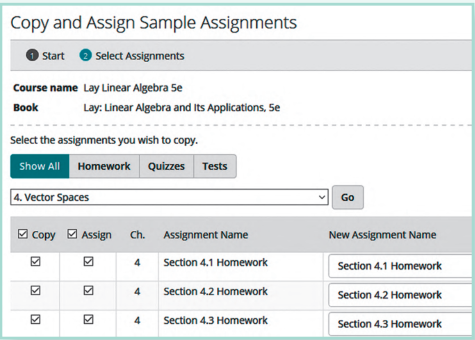
Supporting Instruction

MyLab Math provides resources to help you assess and improve student results and unparalleled flexibility to create a course tailored to you and your students.

PowerPoint® Lecture Slides

Fully editable PowerPoint slides are available for all sections of the text. The slides include definitions, theorems, examples and solutions. When used in the classroom, these slides allow the instructor to focus on teaching, rather than writing on the board. PowerPoint slides are available to students (within the Video and Resource Library in MyLab Math) so that they can follow along.





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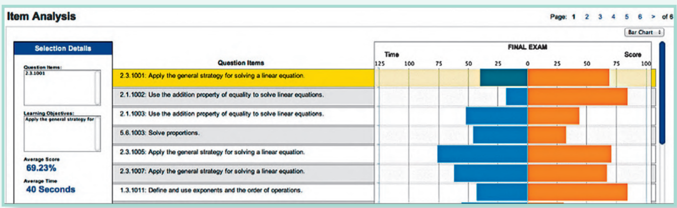
Sample Assignments are crafted to maximize student performance in the course. They make course set-up easier by giving instructors a starting point for each section.

Comprehensive Gradebook

The gradebook includes enhanced reporting functionality, such as item analysis and a reporting dashboard to enable you to efficiently manage your course. Student performance data are presented at the class, section, and program levels in an accessible, visual manner so you'll have the information you need to keep your students on track.



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Resources for

Success

Instructor Resources

Online resources can be downloaded from MyLab Math or from www.pearsonglobaleditions.com.

Instructor's Solution Manual

Includes fully worked solutions to all exercises in the text and teaching notes for many sections.

PowerPoint® Lecture Slides

These fully editable lecture slides are available for all sections of the text.

Instructor's Technology Manuals

Each manual provides detailed guidance for integrating technology throughout the course, written by faculty who teach with the software and this text. Available For MATLAB, Maple, Mathematica, and Texas Instruments graphing calculators.

TestGen®

TestGen (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.

Student Resources

Additional resources to enhance student success. All resources can be downloaded from MyLab Math.

Study Guide

Provides detailed worked-out solutions to every third odd-numbered exercise. Also, a complete explanation is provided whenever an odd-numbered writing exercise has a Hint in the answers. Special subsections of the *Study Guide* suggest how to master key concepts of the course. Frequent "Warnings" identify common errors and show how to prevent them. MATLAB boxes introduce commands as they are needed. Appendixes in the *Study Guide* provide comparable information about Maple, Mathematica, and TI graphing calculators. Available within MyLab math.

Getting Started with Technology

A quick-start guide for students to the technology they may use in this course. Available for MATLAB, Maple, Mathematica, or Texas Instrument graphing calculators. Downloadable from MyLab Math.

Projects

Exploratory projects, written by experienced faculty members, invite students to discover applications of linear algebra.

A Note to Students

This course is potentially the most interesting and worthwhile undergraduate mathematics course you will complete. In fact, some students have written or spoken to us after graduation and said that they still use this text occasionally as a reference in their careers at major corporations and engineering graduate schools. The following remarks offer some practical advice and information to help you master the material and enjoy the course.

In linear algebra, the *concepts* are as important as the *computations.* The simple numerical exercises that begin each exercise set only help you check your understanding of basic procedures. Later in your career, computers will do the calculations, but you will have to choose the calculations, know how to interpret the results, analyze whether the results are reasonable, then explain the results to other people. For this reason, many exercises in the text ask you to explain or justify your calculations. A written explanation is often required as part of the answer. If you are working on questions in MyLab Math, keep a notebook for calculations and notes on what you are learning. For odd-numbered exercises in the textbook, you will find either the desired explanation or at least a good hint. You must avoid the temptation to look at such answers before you have tried to write out the solution yourself. Otherwise, you are likely to think you understand something when in fact you do not.

To master the concepts of linear algebra, you will have to read and reread the text carefully. New terms are in boldface type, sometimes enclosed in a definition box. A glossary of terms is included at the end of the text. Important facts are stated as theorems or are enclosed in tinted boxes, for easy reference. We encourage you to read the Preface to learn more about the structure of this text. This will give you a framework for understanding how the course may proceed.

In a practical sense, linear algebra is a language. You must learn this language the same way you would a foreign language—with daily work. Material presented in one section is not easily understood unless you have thoroughly studied the text and worked the exercises for the preceding sections. Keeping up with the course will save you lots of time and distress!

Numerical Notes

We hope you read the Numerical Notes in the text, even if you are not using a computer or graphing calculator with the text. In real life, most applications of linear algebra involve numerical computations that are subject to some numerical error, even though that error may be extremely small. The Numerical Notes will warn you of potential difficulties in using linear algebra later in your career, and if you study the notes now, you are more likely to remember them later.

If you enjoy reading the Numerical Notes, you may want to take a course later in numerical linear algebra. Because of the high demand for increased computing power, computer scientists and mathematicians work in numerical linear algebra to develop faster and more reliable algorithms for computations, and electrical engineers design faster and smaller computers to run the algorithms. This is an exciting field, and your first course in linear algebra will help you prepare for it.

Study Guide

To help you succeed in this course, we suggest that you use the *Study Guide* available in MyLab Math. Not only will it help you learn linear algebra, it also will show you how to study mathematics. At strategic points in your textbook, marginal notes will remind you to check that section of the *Study Guide* for special subsections entitled "Mastering Linear Algebra Concepts." There you will find suggestions for constructing effective review sheets of key concepts. The act of preparing the sheets is one of the secrets to success in the course, because you will construct *links between ideas.* These links are the "glue" that enables you to build a solid foundation for learning and *remembering* the main concepts in the course.

The *Study Guide* contains a detailed solution to more than a third of the odd-numbered exercises, plus solutions to all odd-numbered writing exercises for which only a hint is given in the Answers section of this book. The *Guide* is separate from the text because you must learn to write solutions by yourself, without much help. (We know from years of experience that easy access to solutions in the back of the text slows the mathematical development of most students.) The *Guide* also provides warnings of common errors and helpful hints that call attention to key exercises and potential exam questions.

If you have access to technology—MATLAB, Octave, Maple, Mathematica, or a TI graphing calculator—you can save many hours of homework time. The *Study Guide* is your "lab manual" that explains how to use each of these matrix utilities. It introduces new commands when they are needed. You will also find that most software commands you might use are easily found using an online search engine. Special matrix commands will perform the computations for you!

What you do in your first few weeks of studying this course will set your pattern for the term and determine how well you finish the course. Please read "How to Study Linear Algebra" in the *Study Guide* as soon as possible. Many students have found the strategies there very helpful, and we hope you will, too.

1 Linear Equations in Linear Algebra

Introductory Example

LINEAR MODELS IN ECONOMICS AND ENGINEERING

It was late summer in 1949. Harvard Professor Wassily Leontief was carefully feeding the last of his punched cards into the university's Mark II computer. The cards contained information about the U.S. economy and represented a summary of more than 250,000 pieces of information produced by the U.S. Bureau of Labor Statistics after two years of intensive work. Leontief had divided the U.S. economy into 500 "sectors," such as the coal industry, the automotive industry, communications, and so on. For each sector, he had written a linear equation that described how the sector distributed its output to the other sectors of the economy. Because the Mark II, one of the largest computers of its day, could not handle the resulting system of 500 equations in 500 unknowns, Leontief had distilled the problem into a system of 42 equations in 42 unknowns.

Programming the Mark II computer for Leontief's 42 equations had required several months of effort, and he was anxious to see how long the computer would take to solve the problem. The Mark II hummed and blinked for 56 hours before finally producing a solution. We will discuss the nature of this solution in Sections 1.6 and 2.6.

Leontief, who was awarded the 1973 Nobel Prize in Economic Science, opened the door to a new era in mathematical modeling in economics. His efforts at Harvard in 1949 marked one of the first significant uses of computers to analyze what was then a large-scale mathematical model. Since that time, researchers in many other fields have employed computers to analyze mathematical models. Because of the massive amounts of data involved, the models are usually *linear*; that is, they are described by *systems of linear equations*.

The importance of linear algebra for applications has risen in direct proportion to the increase in computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Computer science is thus intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations.

Scientists and engineers now work on problems far more complex than even dreamed possible a few decades ago. Today, linear algebra has more potential value for students in many scientific and business fields than any other undergraduate mathematics subject! The material in this text provides the foundation for further work in many interesting areas. Here are a few possibilities; others will be described later.

- *Oil exploration.* When a ship searches for offshore oil deposits, its computers solve thousands of separate systems of linear equations *every day*. The seismic data for the equations are obtained from underwater shock waves created by explosions from air guns. The waves bounce off subsurface rocks and are measured by geophones attached to mile-long cables behind the ship.

- *Linear programming.* Many important management decisions today are made on the basis of linear programming models that use hundreds of variables. The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.
- *Electrical networks.* Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software relies on linear algebra techniques and systems of linear equations.
- *Artificial intelligence.* Linear algebra plays a key role in everything from scrubbing data to facial recognition.
- *Signals and signal processing.* From a digital photograph to the daily price of a stock, important information is recorded as a signal and processed using linear transformations.
- *Machine learning.* Machines (specifically computers) use linear algebra to learn about anything from online shopping preferences to speech recognition.

Systems of linear equations lie at the heart of linear algebra, and this chapter uses them to introduce some of the central concepts of linear algebra in a simple and concrete setting. Sections 1.1 and 1.2 present a systematic method for solving systems of linear equations. This algorithm will be used for computations throughout the text. Sections 1.3 and 1.4 show how a system of linear equations is equivalent to a *vector equation* and to a *matrix equation*. This equivalence will reduce problems involving linear combinations of vectors to questions about systems of linear equations. The fundamental concepts of spanning, linear independence, and linear transformations, studied in the second half of the chapter, will play an essential role throughout the text as we explore the beauty and power of linear algebra.

1.1 Systems of Linear Equations

A **linear equation** in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where b and the **coefficients** $a_1, \dots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and exercises, n is normally between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2\left(\sqrt{6} - x_1\right) + x_3$$

are both linear because they can be rearranged algebraically as in equation (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say, $x_1, \dots, x_n$. An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 - 4x_3 &= -7 \end{aligned} \tag{2}$$

A **solution** of the system is a list $(s_1, s_2, \dots, s_n)$ of numbers that makes each equation a true statement when the values $s_1, \dots, s_n$ are substituted for $x_1, \dots, x_n$, respectively. For instance, $(5, 6.5, 3)$ is a solution of system (2) because, when these values are substituted in (2) for x_1, x_2, x_3, respectively, the equations simplify to $8 = 8$ and $-7 = -7$.

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

The graphs of these equations are lines, which we denote by ℓ_1 and ℓ_2. A pair of numbers (x_1, x_2) satisfies *both* equations in the system if and only if the point (x_1, x_2) lies on both ℓ_1 and ℓ_2. In the system above, the solution is the single point $(3, 2)$, as you can easily verify. See Figure 1.

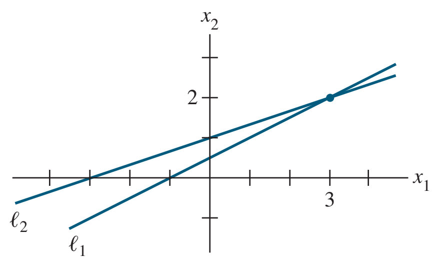


FIGURE 1 Exactly one solution.

Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence "intersect" at every point on the line. Figure 2 shows the graphs that correspond to the following systems:

$$\text{(a)}\quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned} \qquad\qquad \text{(b)}\quad \begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

Figures 1 and 2 illustrate the following general fact about linear systems, to be verified in Section 1.2.

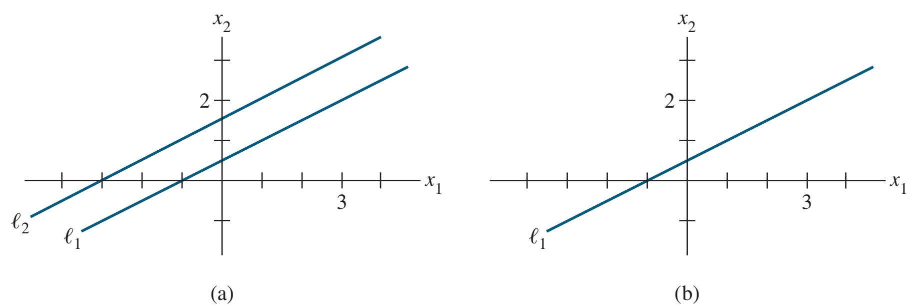


FIGURE 2 (a) No solution. (b) Infinitely many solutions.

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned} \tag{3}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the **coefficient matrix** (or **matrix of coefficients**) of the system (3), and the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \tag{4}$$

is called the **augmented matrix** of the system. (The second row here contains a zero because the second equation could be written as $0 \cdot x_1 + 2x_2 - 8x_3 = 8$.) An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the respective right sides of the equations.

The **size** of a matrix tells how many rows and columns it has. The augmented matrix (4) above has 3 rows and 4 columns and is called a 3×4 (read "3 by 4") matrix. If m and n are positive integers, an $\boldsymbol{m \times n}$ **matrix** is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.) Matrix notation will simplify the calculations in the examples that follow.

Solving a Linear System

This section and the next describe an algorithm, or a systematic procedure, for solving linear systems. The basic strategy is *to replace one system with an equivalent system (that is one with the same solution set) that is easier to solve.*

Roughly speaking, use the x_1 term in the first equation of a system to eliminate the x_1 terms in the other equations. Then use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until you finally obtain a very simple equivalent system of equations.

Three basic operations are used to simplify a linear system: Replace one equation by the sum of itself and a multiple of another equation, interchange two equations, and multiply all the terms in an equation by a nonzero constant. After the first example, you will see why these three operations do not change the solution set of the system.

EXAMPLE 1 Solve system (3).

SOLUTION The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \qquad - 5x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Keep x_1 *in the first equation and eliminate it from the other equations.* To do so, add -5 times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

$$\begin{array}{r|r} -5 \cdot [\text{equation 1}] & -5x_1 + 10x_2 - 5x_3 = 0 \\ + [\text{equation 3}] & 5x_1 \qquad\quad - 5x_3 = 10 \\ \hline [\text{new equation 3}] & 10x_2 - 10x_3 = 10 \end{array}$$

The result of this calculation is written in place of the original third equation:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Now, multiply equation 2 by $\frac{1}{2}$ in order to obtain 1 as the coefficient for x_2. (This calculation will simplify the arithmetic in the next step.)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Use the x_2 in equation 2 to eliminate the $10x_2$ in equation 3. The "mental" computation is

$$\begin{array}{r|r} -10 \cdot [\text{equation 2}] & -10x_2 + 40x_3 = -40 \\ + [\text{equation 3}] & 10x_2 - 10x_3 = 10 \\ \hline [\text{new equation 3}] & 30x_3 = -30 \end{array}$$

The result of this calculation is written in place of the previous third equation (row):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Now, multiply equation 3 by $\frac{1}{30}$ in order to obtain 1 as the coefficient for x_3. (This calculation will simplify the arithmetic in the next step.)

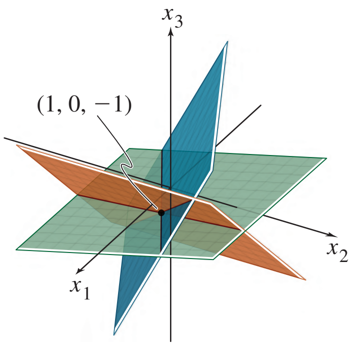
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The new system has a *triangular* form (the intuitive term *triangular* will be replaced by a precise term in the next section):

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Eventually, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 in equation 3 first, to eliminate the $-4x_3$ and $+x_3$ terms in equations 2 and 1. The two "mental" calculations are

$$\begin{array}{rr} 4 \cdot [\text{equation } 3] & 4x_3 = -4 \\ +\ [\text{equation } 2] & x_2 - 4x_3 = 4 \\ \hline [\text{new equation } 2] & x_2 = 0 \end{array} \qquad \begin{array}{rr} -1 \cdot [\text{equation } 3] & -x_3 = 1 \\ +\ [\text{equation } 1] & x_1 - 2x_2 + x_3 = 0 \\ \hline [\text{new equation } 1] & x_1 - 2x_2 = 1 \end{array}$$



Each of the original equations determines a plane in three-dimensional space. The point $(1, 0, -1)$ lies in all three planes.

It is convenient to combine the results of these two operations:

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Now, having cleaned out the column above the x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3, there is now no arithmetic involving x_3 terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The work is essentially done. It shows that the only solution of the original system is $(1, 0, -1)$. However, since there are so many calculations involved, it is a good practice to check the work. To verify that $(1, 0, -1)$ *is* a solution, substitute these values into the left side of the original system, and compute:

$$\begin{aligned} 1(1) - 2(0) + 1(-1) &= 1 - 0 - 1 = 0 \\ 2(0) - 8(-1) &= \phantom{1 -{}} 0 + 8 = 8 \\ 5(1) - 5(-1) &= 5 + 5 = 10 \end{aligned}$$

The results agree with the right side of the original system, so $(1, 0, -1)$ is a solution of the system. ■

Example 1 illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the following operations on the augmented matrix.

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.[1]
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

[1] A common paraphrase of row replacement is "Add to one row a multiple of another row."